# Comparative Study of Sampling-based Multistage Stochastic Linear Programming Algorithms


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## 1. Multistage Stochastic Linear Programs

We consider a class of **multistage stochastic linear programming** (MSLP) problem

$$\begin{aligned}\min_{x_0}\ & d_0^\top x_0 + \mathbb{E}_{\tilde\omega_1}\left[\min_{x_1} d_1^\top x_1 + \mathbb{E}_{\tilde\omega_2}\left[\cdots + \mathbb{E}_{\tilde\omega_T}\left[\min_{x_T} d_T^\top x_T\right]\right]\right],\\ \text{s.t. } & D_0 x_0 = r_0,\\ & C_1 x_0 + D_1 x_1 = r_1(\tilde\omega_1),\\ & \vdots\\ & C_T x_{T-1} + D_T x_T = r_T(\tilde\omega_T),\\ & x_t \geq 0,\ t = 0,1,\ldots,T.\end{aligned} \tag{1}$$

- The cost vector $d_t$, transfer matrix $C_t$ and recourse matrix $D_t$ are known.
- Random variables $\{\tilde\omega_t\}_{t=1}^T$ are stagewise independent with finite support.
- The right-hand side vector $r_t$ is affected by $\tilde\omega_t$ with $t \geq 1$.
- The feasible region $\mathcal{X}_t(x_{t-1}, \omega_t) := \{x \geq 0 \mid D_t x = r_t(\omega_t) - C_t x_{t-1}\}$ is assumed to be nonempty and compact for all $x_{t-1}$ and $\omega_t$.

**Dynamic Programming Form**

$$h_t(x_{t-1}, \omega_t) := \min\left\{d_t^\top x_t + H_{t+1}(x_t) \mid x_t \in \mathcal{X}_t(x_{t-1}, \omega_t)\right\}, \tag{2}$$

where recourse function $H_{t+1}(x_t) := \mathbb{E}_{\tilde\omega_{t+1}}[h_{t+1}(x_t, \tilde\omega_{t+1})]$ and $H_{T+1} = 0$.

## 2. Basic Feasible Policy (BFP)

**Idea**: Obtain a suboptimal solution to a linear program

$$\text{minimize } l(x) \text{ subject to } Dx \leq r,$$

from the discovered index set $\mathcal{C}$ and the mapping defined below

$$\arg\min\{l(x) \mid Dx \leq r, x_{\mathbb{B}} = D_{\mathbb{A},\mathbb{B}}^{-1} r_{\mathbb{A}}, x_{\bar{\mathbb{B}}} = \mathbf{0}, \forall(\mathbb{A},\mathbb{B}) \in \mathcal{C}\},$$

where $\mathbb{A}$ and $\mathbb{B}$ denote the active constraints and basic variable indices, respectively, and $\bar{\cdot}$ represents the complement.

**Toy Example**

$$Dx \leq r$$

$$\left(\begin{array}{ccccc|c} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & r_1\\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & r_2\\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & r_3\end{array}\right)$$

$\mathbb{A} = \{1,3\}$ $\mathbb{B} = \{1,4\}$

$$\begin{pmatrix} x_1\\ x_4\end{pmatrix} = \begin{pmatrix} d_{11} & d_{14}\\ d_{31} & d_{34}\end{pmatrix}^{-1}\begin{pmatrix} r_1\\ r_3\end{pmatrix}$$

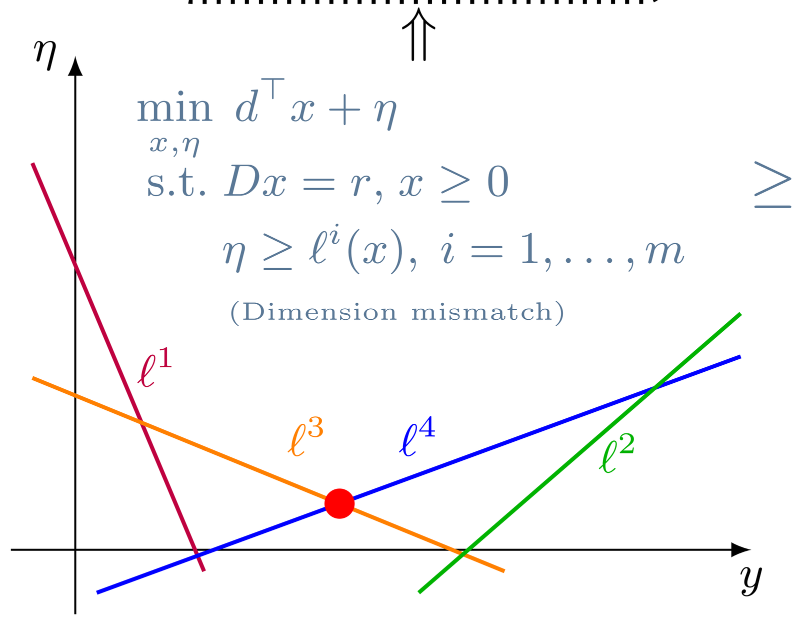
$\Downarrow$

If $d_{21}x_1 + d_{24}x_4 \leq r_2$, then $(x_1, 0, 0, x_4, 0)$ might be selected.

**How to design BFP for SDDP to identify the prox-center?**

Cut

$\Uparrow$




$$\begin{aligned}\min_{x,\eta}\ & d^\top x + \eta\\ \text{s.t. } & Dx = r,\ x \geq 0\\ & \eta \geq \ell^i(x),\ i = 1,\ldots,m\end{aligned}$$

(Dimension mismatch)

$\geq$

**Optimal basis for BFP**

$\Uparrow$

$$\begin{aligned}\min_{x}\ & d^\top x + \ell^{i_\star}(x)\\ \text{s.t. } & Dx = r,\ x \geq 0\end{aligned}$$

Q: How to choose $i_\star$?

A1: Latest strategy

A2: Active strategy

$\Downarrow$

Cut or minorant

## 3. Sampling-based Approaches

**Algorithmic Sketch**

**Sampling**: Simulate a sample-path $\{\omega_t^k\}_{t=1}^T$ randomly.

$\downarrow$

**Prediction pass**: Identify the incumbent sequence $\{\hat{x}_t^k\}_{t=0}^{T-1}$ using the descent condition at $t = 0$ or BFP at $t = 1, \ldots, T-1$.

$\downarrow$

**Forward pass**: Set $\sigma \geq 0$ and identify $\{x_t^k\}_{t=0}^{T-1}$ via solving

$$x_t^k = \underset{x \in \mathcal{X}_t(x_{t-1}^{k+1}, \omega_t^k)}{\arg\min}\ d_t^\top x + H_{t+1}^k(x) + \frac{\sigma}{2}\|x - \hat{x}_t^k\|^2,\ t = 0,\ldots,T-1.$$

$\downarrow$

**Backward recursion**: Update the recourse approximate $H_t^{k+1}$ by adding new cuts or minorants $(\alpha_t^k, \beta_t^k)$ and $(\hat\alpha_t^k, \hat\beta_t^k)$ at $t = T, \ldots, 1$.

$\downarrow$

**Stopping criterion**: Exit if the estimated lower bound lies within the statistical upper confidence interval.

**Stochastic Minorants**

The sample average $H_t^k(x)$ admits a lower approximation of the form $\alpha_t^k + \langle \beta_t^k, x\rangle$, where

$$\alpha_t^k = \sum_{\omega_t \in \Omega_t^k} \mathbb{P}(\omega_t)\alpha_t^k(\omega_t) \text{ and } \beta_t^k = \sum_{\omega_t \in \Omega_t^k} \mathbb{P}(\omega_t)\beta_t^k(\omega_t) \text{ with the observations } \Omega_t^k.$$

(i) Case $\omega_t^k$: $\alpha_t^k(\omega_t^k) = \alpha_{t+1}^k + \langle r_t(\omega_t^k), \pi_t^k\rangle$ and $\beta_t^k(\omega_t^k) = -C_t^\top \pi_t^k$ with $\pi_t^k$ being the maximizer of

$$\max_{D_t^\top \pi \leq d_t + \beta_{t+1}^k} \alpha_{t+1}^k + \langle r_t(\omega_t^k) - C_t x_{t-1}^k, \pi\rangle \overset{\text{(by duality)}}{\leq} \min_{D_t x = r_t(\omega_t^k) - C_t x_{t-1}^k, x \geq 0} d_t^\top x + \alpha_{t+1}^k + \langle \beta_{t+1}^k, x\rangle. \tag{3}$$

Furthermore, one may update $\Pi_t^k(\omega_t^k) \leftarrow \Pi_t^{k-1}(\omega_t^k) \cup \{(\alpha_t^k(\omega_t^k), \beta_t^k(\omega_t^k))\}$ if necessary.

(ii) Case $\omega_t \in \Omega_t^k \setminus \{\omega_t^k\}$:

$$[\alpha_t^k(\omega_t), \beta_t^k(\omega_t)] = \underset{(\alpha_t^i, \beta_t^i) \in \Pi_t^k(\omega_t)}{\arg\max} \left(\frac{i}{k}\right)^{T+1-t}\left(\alpha_t^i + \langle \beta_t^i, x_{t-1}^k\rangle\right) + \left(1 - \left(\frac{i}{k}\right)^{T+1-t}\right)L. \tag{4}$$

An analogous procedure is performed on the incumbent sequence $\{\hat{x}_t^k\}_{t=1}^T$.

$$\mathcal{J}_t^k \leftarrow \left[\left(\frac{k-1}{k}\right)^{T+1-t}\mathcal{J}_t^{k-1} + \left(1 - \left(\frac{k-1}{k}\right)^{T+1-t}\right)(L, \mathbf{0})\right] \cup \left\{(\alpha_t^k, \beta_t^k), (\hat\alpha_t^k, \hat\beta_t^k)\right\}.$$

**Stochastic Dual Dynamic Programming**

- Known $\mathbb{P}(\omega_t)$
- Deterministic cuts

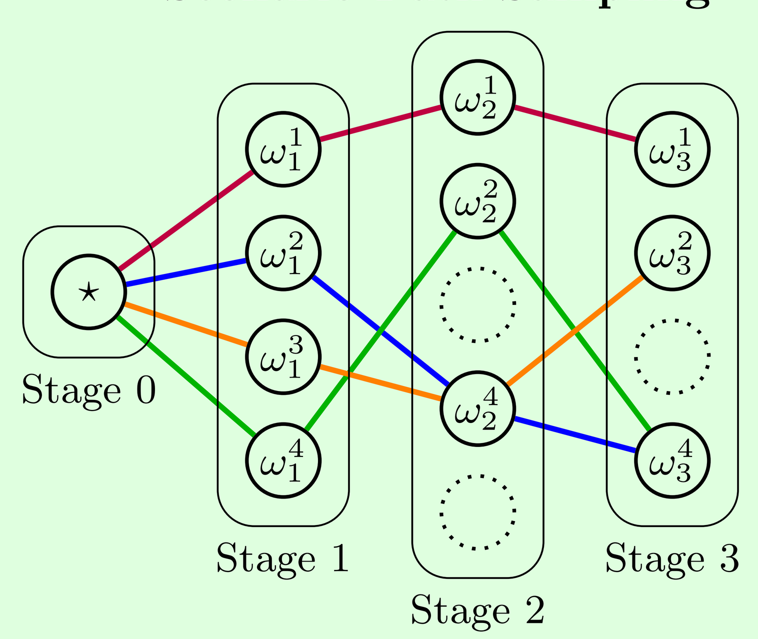
**Regularization Technique**

- Prox-center $\{\hat{x}_t^k\}_{t=1}^T$ by BFP
- Quadratic regularization

**Stochastic Dynamic Linear Programming**

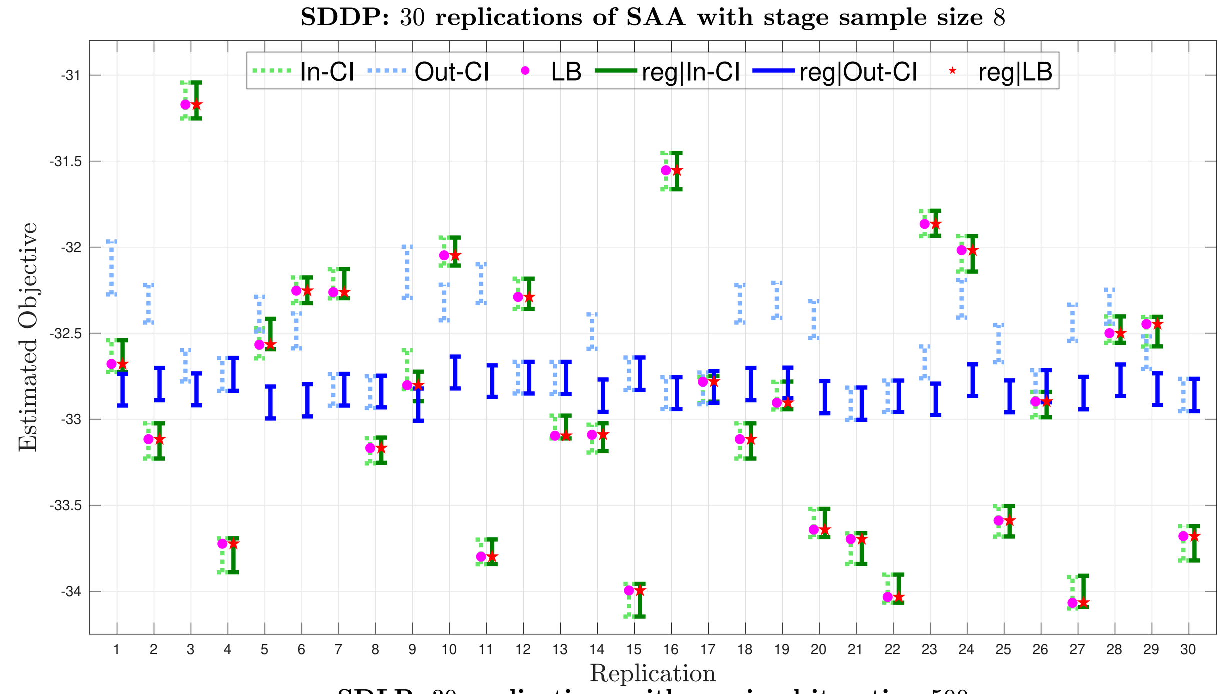
- Known uniform lower bound $L$
- Stochastic minorants along $\{x_t^k\}_{t=1}^T$ & $\{\hat{x}_t^k\}_{t=1}^T$

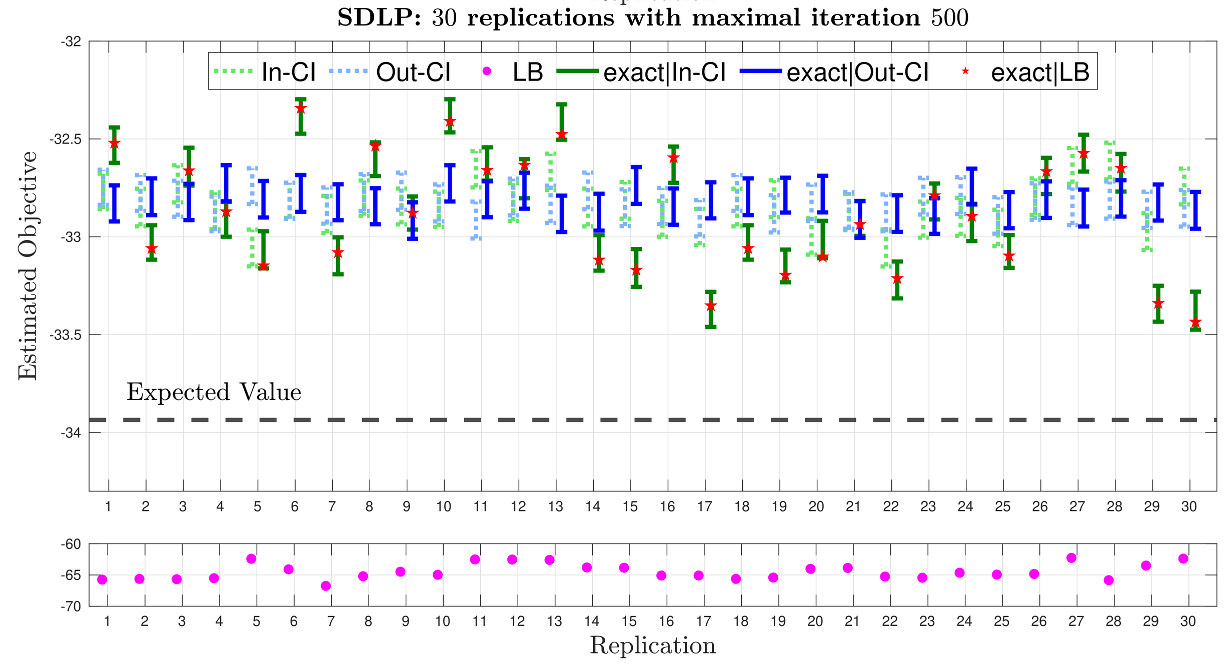
**Scenario-Path Sampling**




## 4. Numerical Experiments

- PLTEXPA7: 7 stages with 16 scenarios at each non-root stage.
- 95% statistical confidence interval (CI) based on 2000 sample-paths.




- BFP based regularization improves solution quality.
- Compared to SDDP with fixed sample size, exact SDLP yields better solutions while the lower bound of SDLP is weak.

## 5. Conclusions

**Summary**: This study presented a regularized SDDP that employs BFP to identify the prox-center and investigated sampling-based (SDDP & SDLP) approaches for solving MSLP problems numerically.

**Future Directions**:

- Convergence analysis of the regularized SDDP.
- Strategies for discarding poor cuts/minorants in later iterations.
- Improving lower bounds & exploration of stopping rules for SDLP.
- Implementation of multicuts.

## References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This research is funded by Advanced Scientific Computing Research Program at the Office of Science, US Department of Energy under Grant # DE-SC0023361.